A MATHEMATICAL MODEL FOR OPTIMIZATION OF CATCH QUOTA MANAGEMENT IN MIXED FISHERIES

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ABSTRACT. Discard of fish remains to be one of the most important problems to solve in fisheries management worldwide. The EU Commission proposed in July 2011 a revised Common Fisheries Policy (CFP) based on full catch accountability and "all fish landed". This i.a. imply that all catches must count against a quota and that fisheries must stop when quotas are exhausted. In mixed fisheries the consequence is that the fishery for all species is stopped as soon as one of the quotas is exhausted. The effect of this management approach on the total uptake of all TAC/quotas depends i.a. on the relative strength of the species in the mixed fishery, catchability and the vessels and fleets ability to target their fishery operation against the specific species components in the mixed fishery. Furthermore it depends on the allocation of TAC's between EU member states (Relative Stability), the national quota allocation and possible mechanisms of quota transferability between vessels.

The catch quota managemant (CQM) optimization model developed in this paper allow the manager and the industry to assess consequences for total quota uptake based on variations in the above mentioned factors.

1. INTRODUCTION

Utilizing the marine food potential to its full requires a high degree of knowledge, skill and technology to overcome the variability and complexities in the harvesting.

At present, management worldwide is based on a "command and control" approach setting more and more detailed rules and controls for the fishery. An increasing focus of management has been devoted to reducing the gap between registered catches (landings) and total catches. At the same time management accept or even require fishermen to discard fish once it is caught. In mixed fisheries, it is commonly accepted that a fishery continue until all quotas have been exhausted with the result that catches from the least plentiful species are discarded until the quota for the most plentiful species has been reached.

13th July 2011 the EU Commission in its proposal [1] for a revised Common Fisheries Policy (CFP) stated that the new policy should be based on all fish landed. This entails a transformation to a policy where all fish caught count on the quota and where all fish caught must be covered by a quota. The principle of full accountability is supported by the principle of full documentation as the Commission proposes that fishing vessels must be equipped with electronic monitoring (CCTV, sensorsystems and E-log).

The proposal is i.a. based on extensive trials with Catch Quota Management (CQM) and full documentation in Denmark, Germany and UK.

CQM entails that a fishery require a catch quota for all commercial species caught in that fishery. When the least plentiful quota in a mixed fishery—the "choke species"—is exhausted, the fishery must stop.

While CQM stops excess fishing mortality due to discarding it also bring forward the problem for fishermen to fully utilize the quotas they have been allocated in mixed fisheries. Failing to utilize the plentiful quota because of exhaustion of the choke species will result in loss of income.

A number of factors are influencing the balances of optimal quota utilization in mixed fisheries.

Key words and phrases. Catch Quota Management (CQM); Common Fisheries Policy (CFP); EU fisheries policy; mathematical modelling; optimization; result based management.

- *Catchability:* Are all species in a mixed fishery catchable to a degree that reflect their abundance, hence the TAC's set? And are quotas set on basis of sound biological evidence?
  TAC's are broken down on national quotas and again on fleet and vessel level. How does this affect optimal quota utilization? And do the quotas set reflect the actual stock situation?
- *Targeting efficiency:* To what extent can the individual vessel target the individual stocks in a mixed fishery by means of planning, method and technology?
- *Transferability:* To what extent will transferable quota systems allow for vessels and fleets to fully utilize quotas?

The model developed in Section 2 and 3 will make it possible when fixing TAC/quotas to assess consequences for total quota uptake of varying relationships between the least and the most plentiful species in mixed fisheries, of changes in vessels targeting ability and of changes in quota transferability.

2. THE SINGLE VESSEL CASE

In this section, we consider a fishery in which a single vessel[1] catches two types of fish, called *type 1* and *type 2*, in some combination. We assume in this section that the vessel is completely isolated in the sense that fish quotas cannot be leased or bought from others vessels. We shall get rid of this restrictive assumption in Section 3.

2.1 Example. Some typical type 1/type 2 fish combinations are:
- cod/haddock (Scottish fleet)
- cod/nephrops (Danish fleet)
- sole/plaice (Dutch fleet)

Our goal in this section is, in a nutshell, to determine the type 1/type 2 catch composition that results in the maximal income given that the vessel's targetting ability may not allow it to fully utilize both catch quotas in the mixed fishery. To find the optimal catch composition, we must first give a mathematical description of the situation we are considering.

Assumptions. The market prices (kr/kg or kilokr/ton) of the two types of fish determine the market data:

Market Data	
Market price of type 1 fish	p_1
Market price of type 2 fish	p_2

2.2 Example. Prices obtained for landed catches vary a lot. Suggestive levels are:

$$\begin{aligned} p_{\text{cod}} &= 20 \text{ (kilokr/ton)} \\ p_{\text{haddock}} &= 14 \text{ (kilokr/ton)} \\ p_{\text{nephrops}} &= 55 \text{ (kilokr/ton)} \\ p_{\text{sole}} &= 85 \text{ (kilokr/ton)} \\ p_{\text{plaice}} &= 10 \text{ (kilokr/ton)} \end{aligned}$$

The vessel under consideration catches fish of type 1 and type 2 in some combination determined by the *catch composition parameter*, $c \in [0,1]$, whose value the vessel can partially influence by using various types of equipment. The catch composition parameter is simply defined as *the percentage of type 2 fish in the total catch*. Thus we have:

$$\begin{array}{lcc} \text{The percentage of type 1 fish in the total catch is} & : & 1-c \\ \text{The percentage of type 2 fish in the total catch is} & : & c \end{array}$$

[1] The single vessel under consideration could be a specific vessel in some fleet, however, it could also be an "abstract" vessel representing a group of actual vessels fishing under similar conditions. Another possibility is to think of the vessel as the average of an entire fleet.

2.3 Example. The value $c = 0.55$ corresponds to a type 1/type 2 catch composition of $45\%/55\%$.

The vessel's fishing equipment allows it to vary the catch composition parameter in a certain range from a *lower catch composition bound*, l, to an *upper catch composition bound*, u.

2.4 Example. The values $l = 0.5$ and $u = 0.6$ correspond to the situation where the vessel may vary its type 1/type 2 catch composition in the range from $50\%/50\%$ to $40\%/60\%$.

For the vessel in question, we assume that the bounds l and u are given. The value of the catch composition parameter $c \in [l,u]$ is then to be determined such that the vessel maximizes its income. The vessel's fishery is bounded by its *type 1 fish quota*, Q_1, and its *type 2 fish quota*, Q_2 (in tonnes). In Catch Quota Management, the vessel must stop fishing once it reaches one of these two quotas.

2.5 Example. Examples of member state quotas (in tonnes) are:

Scottish fleet:	$Q_{\text{cod}} = 11000$	and	$Q_{\text{haddock}} = 21000$
Danish fleet:	$Q_{\text{cod}} = 4000$	and	$Q_{\text{nephrops}} = 4000$
Dutch fleet:	$Q_{\text{sole}} = 10000$	and	$Q_{\text{plaice}} = 20000$

Thus, a single vessel in the Scottish fleet may have quotas $Q_{\text{cod}} = 11$ and $Q_{\text{haddock}} = 21$ (tonnes).

We summarize the mathematical data describing the vessel under consideration:

Vessel Data	
Type 1 fish quota	Q_1
Type 2 fish quota	Q_2
Lower catch composition bound	l
Upper catch composition bound	u
Catch composition parameter	c

Here Q_1, Q_2, l, and u are given, whereas c is to be (optimally) determined in the range $[l,u]$.

The Mathematical Model. Denote by t_1 and t_2 the number of tonnes of fish of type 1 and type 2 caught by the vessel. In catch quota management, both of these numbers are bounded by the vessel's given quotas, that is,

(i) $$t_1 \leqslant Q_1$$

(ii) $$t_2 \leqslant Q_2$$

The total catch is $t_1 + t_2$ of which t_2, by definition of the catch composition parameter, constitutes the fraction c, that is,

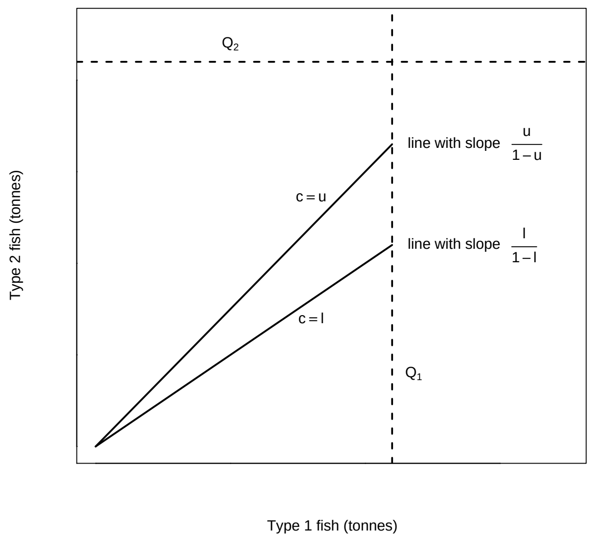
$$\frac{t_2}{t_1 + t_2} = c.$$

Equivalently,

(iii$_c$) $$ct_1 - (1-c)t_2 = 0$$

This is a straight line in a (t_1,t_2) coordinate system with slope $a = \frac{c}{1-c}$ and constant term $b = 0$. Since c can vary in the interval $[l,u]$, the slope of the line in (iii$_c$) can vary in the interval $[\frac{l}{1-l}, \frac{u}{1-u}]$.

The restrictions (i), (ii), and (iii$_c$) are illustrated in the following (t_1,t_2) coordinate system.



The vessel's *income* for catching (and selling) t_1 tonnes of fish of type 1, and t_2 tonnes of fish of type 2 is computed from the market prices:

$$I(t_1,t_2) = p_1 t_1 + p_2 t_2.$$

For each value of the parameter c in the interval $[l,u]$—corresponding to a choice of type 1/type 2 catch composition within the range determined by the vessel's equipment—the maximal income is found by optimizing the income function $I(t_1,t_2)$ (in two variables) subject to the boundary conditions (i), (ii), and (iii$_c$). This optimization problem, in which the third boundary condition depends on the value of c, may be written in short form as follows.

$$(P_c) \qquad \begin{cases} I(t_1,t_2) = p_1 t_1 + p_2 t_2 = \text{Max!} \\ \qquad t_1 \leqslant Q_1 \\ \qquad t_2 \leqslant Q_2 \\ ct_1 - (1-c)t_2 = 0 \end{cases}$$

The optimization problem (P_c) above is an example of a so-called linear programming problem in two variables (t_1,t_2), which—given any specific value of c—is easily solved.

2.6 Example. We consider a whitefish fishery where type 1/type 2 fish are cod/haddock. Prices and quotas are as in Examples 2.2 and 2.5, that is,

$$\begin{array}{ll} p_1 = 20 \qquad & Q_1 = 11 \\ p_2 = 14 \qquad & Q_2 = 21 \end{array}$$

Suppose, as in Example 2.3, that the vessel aims for cod/haddock catch composition of $45\%/55\%$ corresponding to the catch composition parameter $c = 0.55$. To maximize its income, the vessel must solve:

$$(P_{0.55}) \qquad \begin{cases} I(t_1,t_2) = 20t_1 + 14t_2 = \text{Max!} \\ \qquad t_1 \leqslant 11 \\ \qquad t_2 \leqslant 21 \\ 0.55t_1 - 0.45t_2 = 0 \end{cases}$$

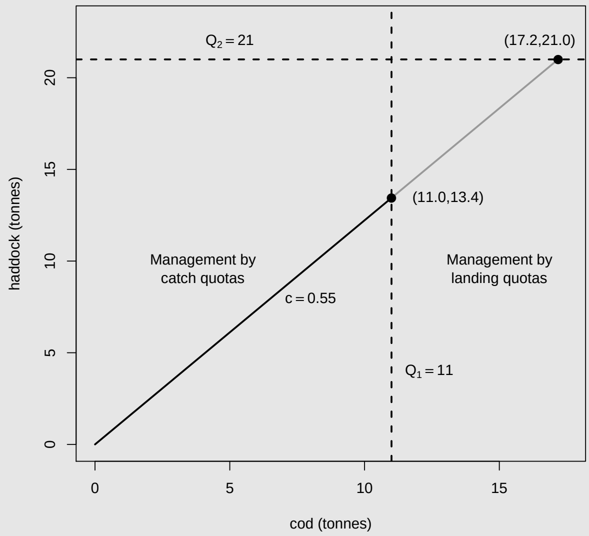
As illustrated below, the solution to this problem is:

$$(t_1^*,t_2^*) \simeq (11,13.4) \qquad \text{(tonnes)}.$$

Note that the ratio of $t_2^* = 13.4$ and the total catch $t_1^* + t_2^* = 24.4$ really is $c = 0.55$, and that the fishery must stop since the cod quota $Q_1 = 11$ has been reached. Therefore, subject to management by *catch quotas*, the income is:

$$I^* = I(t_1^*,t_2^*) = I(11,13.4) = 20 \cdot 11 + 14 \cdot 13.4 \simeq 408 \ \text{(kilokr)}.$$

Example (continued).



In comparison, the present management by *landing quotas* would generate an income of

$$I(11,21) = 20 \cdot 11 + 14 \cdot 21 = 514 \ \ \text{(kilokr)},$$

that is, 106 (kilokr) more than under catch quota management. However, the resulting discards of cods would be $17.2 - 11 = 6.2$ (tonnes), which has a market value of 124 (kilokr).

Thus, under management by landing quotas—and with the given assumptions—one must, in some sense, "throw away" 124 (kilokr) to earn an extra 106 (kilokr).

Recall that our goal is to find the optimal catch composition, that is, the catch composition which results in the maximal income. In view of this, the example above, although illustrative, is not very informative since it requires the catch composition parameter c to be known in advance. To overcome this obstacle, we return to the optimization problem (P_c). For any given c in $[l,u]$, we denote by

$$(t_1^*(c), t_2^*(c))$$

the maximum point of the problem (P_c), and we let

$$I^*(c) = I(t_1^*(c), t_2^*(c)) = p_1 t_1^*(c) + p_2 t_2^*(c)$$

be the associated maximal income.

2.7 Example. Consider the setup of Example 2.6. As seen in loc. cit., one has in this case:

$$(t_1^*(0.55), t_2^*(0.55)) = (11, 13.4) \quad \text{and} \quad I^*(0.55) = 408.$$

With this notation at hand, we can give a precise mathematical formulation of the vessel's income-optimization problem.

2.8 The Single Vessel Optimization Problem. The following data are given:

- The market prices p_1 and p_2.
- The vessel's quotas Q_1 and Q_2.
- The vessel's lower and upper catch composition bounds l and u.

To maximize its income, the vessel must find the value $c^\circ \in [l,u]$ of the catch composition parameter which makes the income $I^*(c)$ as large as possible, that is,

$$I^*(c^\circ) = \max_{c \in [l,u]} I^*(c).$$

This *maximal income* is denoted by I°, and c° is called the *optimal catch composition parameter*. The corresponding *optimal (type 1, type 2) catch composition*, that is, $(t_1^*(c^\circ), t_2^*(c^\circ))$ is denoted by (t_1°, t_2°).

Solution. It takes only straightforward graphical considerations to solve the single vessel optimization problem formulated above. The precise solution presented below, which may seem a bit technical at first sight, will be important in Section 3.

2.9 Solution of the Single Vessel Optimization Problem. Depending on the three cases,

$$\begin{aligned} &\text{Case I:} && \frac{Q_2}{Q_1} < \frac{l}{1-l} \\ &\text{Case II:} && \frac{l}{1-l} \leqslant \frac{Q_2}{Q_1} \leqslant \frac{u}{1-u} \\ &\text{Case III:} && \frac{u}{1-u} < \frac{Q_2}{Q_1} \end{aligned}$$

the optimal catch composition parameter c°, the optimal catch composition (t_1°, t_2°), and the maximal income I° are given as follows.

$$c^\circ = \begin{cases} l & \text{(case I)} \\ \dfrac{Q_2}{Q_1+Q_2} & \text{(case II)} \\ u & \text{(case III)} \end{cases}$$

$$(t_1^\circ, t_2^\circ) = \begin{cases} (\frac{1-l}{l}Q_2, Q_2) & \text{(case I)} \\ (Q_1, Q_2) & \text{(case II)} \\ (Q_1, \frac{u}{1-u}Q_1) & \text{(case III)} \end{cases}$$

$$I^\circ = \begin{cases} (p_1\frac{1-l}{l} + p_2)Q_2 & \text{(case I)} \\ p_1Q_1 + p_2Q_2 & \text{(case II)} \\ (p_1 + p_2\frac{u}{1-u})Q_1 & \text{(case III)} \end{cases}$$

2.10 Example. Consider the setup in Example 2.6, that is, type 1/type 2 is cod/haddock, and

$$\begin{aligned} p_1 &= 20 & Q_1 &= 11 \\ p_2 &= 14 & Q_2 &= 21 \end{aligned}$$

Assume that the vessel's equipment allows it to vary its cod/haddock catch composition in the range from 50%/50% to 40%/60%. As in Example 2.4, this corresponds to the bounds:

$$l = 0.5 \qquad \text{and} \qquad u = 0.6$$

The three cases I, II, and III described above are determind by the numbers

$$\frac{l}{1-l} = \frac{0.5}{1-0.5} = 1.0 \qquad \text{and} \qquad \frac{u}{1-u} = \frac{0.6}{1-0.6} = 1.5$$

Since $\frac{Q_2}{Q_1} = \frac{21}{11} \simeq 1.9 > 1.5$ we are in Case III. Hence, the formulae above give the optimal values:

$$c^\circ = u = 0.6 \quad , \quad (t_1^\circ, t_2^\circ) = (11, 16.5) \quad , \quad I^\circ = 451.$$

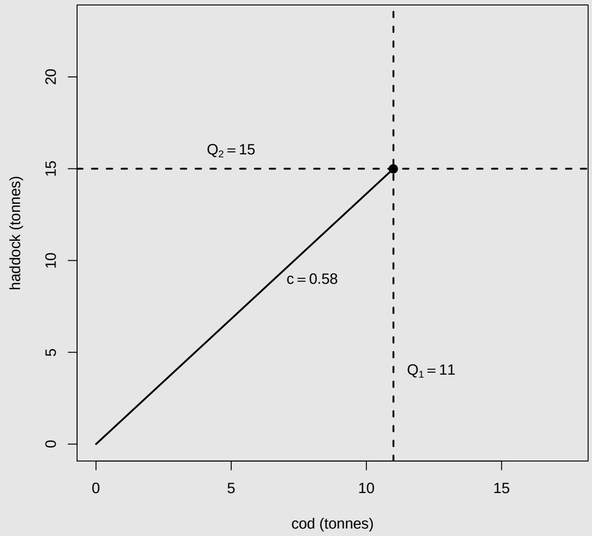
Thus, within the possible range, it is optimal for the vessel to aim for an cod/haddock catch composition of 40%/60%, in which case the fishery stops after having caught $t_1^\circ = 11$ tonnes of cod and $t_2^\circ = 16.5$ tonnes of haddock. The total market value for these fish are $I^\circ = 451$ kilokr.

We note that the catch composition 45%/55% chosen in Example 2.6 is *not* optimal since it only generates an income of 408 (kilokr).

Suppose that next season, the vessel's haddock quota drops from $Q_2 = 21$ to $Q_2 = 15$, say. The vessel must then change its fishery accordingly, indeed, now $\frac{Q_2}{Q_1} = \frac{15}{11} \simeq 1.4$ lies between 1.0 and 1.5 which puts us in Case II. This time, the formulae above give the optimal values:

$$c^\circ = 0.58 \quad , \quad (t_1^\circ, t_2^\circ) = (11, 15) \quad , \quad I^\circ = 430.$$

Example (continued). Hence, it is now optimal for the vessel to aim for an cod/haddock catch composition of $42\%/58\%$, in which case both quotas $Q_1 = 11$ and $Q_2 = 15$ are reached simultaneously.



As implied by the example above, the formulae in our solution of the single vessel optimization problem can also be used to assess the boundaries for the quota relation between type 1 and type 2 fish which will make it possible for the vessel to obtain full quota utilization of both species

3. The Multiple Vessel Case

In this section, we expand on the situation from Section 2 by considering a mixed fishery in which *two* vessels[2], called *vessel A* and *vessel B*, both catch type 1 and type 2 fish in varying combinations. The vessels may may lease[3] quotas from each other in order to optimize their quota portfolios and, in turn, their income. From a mathematical point of view, the leasing aspect complicates the situation considerably compared to the single vessel case studied in Section 2.

Assumptions. In the present situation, the market is defined by the market prices (in kr/kg or kilokr/ton) of the two types of fish, and by the leasing prices (also in kr/kg or kilokr/ton) of the two types of fish quota.

Market Data	
Market price of type 1 fish	p_1
Market price of type 2 fish	p_2
Leasing price of type 1 fish quota	q_1
Leasing price of type 2 fish quota	q_2

[2] The two vessels under consideration could very well be two specific vessels in some fleet. Alternatively, one might think of the two vessels as being "abstract" ones representing some relevant situation. For example, one vessel could represent an actual group of vessels in some fleet fishing under similar conditions, while the other vessel could represent the "residual fleet", that is, the average of the remaining vessels in the fleet.

[3] In real life fishery, quotas may transferred (bought or leased) in some member states. *Buying* of quotas is understood as permanent buying of quota shares, and consequently the quota amount released every year on account of Total Allowable Catches (TACs) is being set accordingly. Buying typically takes place in context of structural changes, as private scrapping investments in new vessels etc. Buying is not considered here. *Leasing* of quotas relate to the amount in tonnes of a given stock leased for the given quota year. Leasing may take place in context of planning the fishery for the entire quota year, or it may relate to daily adaptation of vessel quotas to the development in the fishery—or to cover unforeseen bycatches. The leasing element is considered here as an important tool of flexible quota management.

3.1 Example. Prices for leasing of quotas vary a lot. Suggestive levels are:

$$\begin{aligned} q_{\text{cod}} &= 10 \text{ (kilokr/ton)} \\ q_{\text{haddock}} &= 3 \text{ (kilokr/ton)} \\ q_{\text{nephrops}} &= 3 \text{ (kilokr/ton)} \\ q_{\text{sole}} &= 6 \text{ (kilokr/ton)} \\ q_{\text{plaice}} &= 2 \text{ (kilokr/ton)} \end{aligned}$$

Much of the data defining the two vessels under consideration are the same as described in Section 2, however, each vessel has its own quotas, its own catch composition bounds etc.

Vessel A Data	
Type 1 fish quota	Q_1^{A}
Type 2 fish quota	Q_2^{A}
Lower catch composition bound	l_{A}
Upper catch composition bound	u_{A}
Catch composition parameter	c_{A}
Type 1 fish quota leased from vessel B	x_1
Type 2 fish quota leased from vessel B	x_2

Vessel B Data	
Type 1 fish quota	Q_1^{B}
Type 2 fish quota	Q_2^{B}
Lower catch composition bound	l_{B}
Upper catch composition bound	u_{B}
Catch composition parameter	c_{B}
Type 1 fish quota leased from vessel A	$-x_1$
Type 2 fish quota leased from vessel A	$-x_2$

Since the two vessels may lease quotas from each other, we must consider two additional variables, x_1 and x_2. Here x_1 and x_2 are the number (in tonnes) of type 1 and type 2 fish quotas, respectively, which *vessel A leases from vessel B*. Alternatively, vessel B leases $-x_1$ and $-x_2$ tonnes of type 1 and type 2 fish quota from vessel A. The pair (x_1,x_2) is called a *quota leasing agreement* between the vessels A and B.

3.2 Example. The situation where:

- vessel A leases 10 tonnes of type 1 fish quota *from* vessel B, and
- vessel A leases 15 tonnes of type 2 fish quota *to* vessel B

corresponds to the quota leasing agreement $(x_1,x_2) = (10,-15)$.

The Mathematical Model. Since vessel A can lease at most Q_1^{B} tonnes of type 1 fish quota *from* vessel B one has $x_1 \leqslant Q_1^{\mathrm{B}}$; and since vessel A can lease at most Q_1^{A} tonnes of type 1 fish quota *to* vessel B one has $-x_1 \leqslant Q_1^{\mathrm{A}}$. Combining this with similar considerations for the variable x_2 give the following restrictions:

$$\begin{aligned} -Q_1^{\mathrm{A}} \leqslant x_1 \leqslant Q_1^{\mathrm{B}} \\ -Q_2^{\mathrm{A}} \leqslant x_2 \leqslant Q_2^{\mathrm{B}} \end{aligned}$$

Once a quota leasing agreement (x_1,x_2) has been made, the vessels' quota portfolios change as follows:

- Vessel A's quota portfolio changes from $(Q_1^{\mathrm{A}},Q_2^{\mathrm{A}})$ to $(Q_1^{\mathrm{A}}+x_1,Q_2^{\mathrm{A}}+x_2)$, and
- Vessel B's quota portfolio changes from $(Q_1^{\mathrm{B}},Q_2^{\mathrm{B}})$ to $(Q_1^{\mathrm{B}}-x_1,Q_2^{\mathrm{B}}-x_2)$.

As in Section 2, we denote by $(t_1^{\mathrm{A}},t_2^{\mathrm{A}})$ the number of tonnes of fish of (type 1, type 2) caught by vessel A. Similarly $(t_1^{\mathrm{B}},t_2^{\mathrm{B}})$ denotes the number of tonnes of fish of (type 1, type 2) caught by vessel B. As described in Section 2, the catches for vessels A and B are subject to the conditions:

$$\begin{aligned} t_1^{\mathrm{A}} &\leqslant Q_1^{\mathrm{A}}+x_1 \\ t_2^{\mathrm{A}} &\leqslant Q_2^{\mathrm{A}}+x_2 \\ c_{\mathrm{A}}t_1^{\mathrm{A}}-(1-c_{\mathrm{A}})t_2^{\mathrm{A}} &= 0 \end{aligned} \qquad \text{and} \qquad \begin{aligned} t_1^{\mathrm{B}} &\leqslant Q_1^{\mathrm{B}}-x_1 \\ t_2^{\mathrm{B}} &\leqslant Q_2^{\mathrm{B}}-x_2 \\ c_{\mathrm{B}}t_1^{\mathrm{B}}-(1-c_{\mathrm{B}})t_2^{\mathrm{B}} &= 0 \end{aligned}$$

Since it costs money to lease quotas from the other vessel (dually, one gets paid for leasing quotas to the other vessel), the income functions for the two vessels become more complicated, in fact, they are given by:

$$\begin{aligned} I_{\mathrm{A}}(t_1^{\mathrm{A}},t_2^{\mathrm{A}},x_1,x_2) &= p_1t_1^{\mathrm{A}}+p_2t_2^{\mathrm{A}}-q_1x_1-q_2x_2, \\ I_{\mathrm{B}}(t_1^{\mathrm{B}},t_2^{\mathrm{B}},x_1,x_2) &= p_1t_1^{\mathrm{B}}+p_2t_2^{\mathrm{B}}+q_1x_1+q_2x_2. \end{aligned}$$

We begin by focusing on vessel A. The situation from vessel B's point of view can be described similarly.

For each value of the parameter $c_{\mathrm{A}} \in [l_{\mathrm{A}}, u_{\mathrm{A}}]$—corresponding to a choice of type 1/type 2 catch composition within the range determined by vessel A's equipment—the maximal income for vessel A is found by optimizing the income function $I_{\mathrm{A}}(t_1^{\mathrm{A}}, t_2^{\mathrm{A}}, x_1, x_2)$ subject to the boundary conditions above, i.e.

$$(R^{\mathrm{A}}_{c_{\mathrm{A}}}) \qquad \begin{cases} I_{\mathrm{A}}(t_1^{\mathrm{A}}, t_2^{\mathrm{A}}, x_1, x_2) = p_1 t_1^{\mathrm{A}} + p_2 t_2^{\mathrm{A}} - q_1 x_1 - q_2 x_2 = \text{Max!} \\ t_1^{\mathrm{A}} \leqslant Q_1^{\mathrm{A}} + x_1 \\ t_2^{\mathrm{A}} \leqslant Q_2^{\mathrm{A}} + x_2 \\ c_{\mathrm{A}} t_1^{\mathrm{A}} - (1 - c_{\mathrm{A}}) t_2^{\mathrm{A}} = 0 \\ -Q_1^{\mathrm{A}} \leqslant x_1 \leqslant Q_1^{\mathrm{B}} \\ -Q_2^{\mathrm{A}} \leqslant x_2 \leqslant Q_2^{\mathrm{B}} \end{cases}$$

If the parameter c_{A} is given then $(R^{\mathrm{A}}_{c_{\mathrm{A}}})$ is a so-called linear programming problem in four variables $(t_1^{\mathrm{A}}, t_2^{\mathrm{A}}, x_1, x_2)$, which can be solved using Dantzig's simplex algorithm. However, this is not particularly useful since we do not know the optimal catch composition parameter in advance. To overcome this difficulty, we note that for any fixed choice of quota leasing agreement $(x_1, x_2) \in [-Q_1^{\mathrm{A}}, Q_1^{\mathrm{B}}] \times [-Q_2^{\mathrm{A}}, Q_2^{\mathrm{B}}]$, the problem $(R^{\mathrm{A}}_{c_{\mathrm{A}}})$ reduces to a problem in two variables t_1^{A} and t_2^{A}, namely:

$$\begin{cases} I_{\mathrm{A}}(x_1, x_2)(t_1^{\mathrm{A}}, t_2^{\mathrm{A}}) = p_1 t_1^{\mathrm{A}} + p_2 t_2^{\mathrm{A}} - q_1 x_1 - q_2 x_2 = \text{Max!} \\ t_1^{\mathrm{A}} \leqslant Q_1^{\mathrm{A}} + x_1 \\ t_2^{\mathrm{A}} \leqslant Q_2^{\mathrm{A}} + x_2 \\ c_{\mathrm{A}} t_1^{\mathrm{A}} - (1 - c_{\mathrm{A}}) t_2^{\mathrm{A}} = 0 \end{cases}$$

Note that this is nothing but the single vessel optimization problem which was formulated in 2.8 (the only difference being that Q_i^{A} is replaced by $Q_i^{\mathrm{A}} + x_i$) and solved in 2.9. Thus, supposing that vessel A and vessel B make the quota leasing agreement (x_1, x_2), then vessel A knows how to fish optimally and thus maximize its income. In fact, the following formulae are immediate from the solution found in 2.9.

3.3 Quota Leasing Agreements from the Viewpoint of Vessel A. Suppose that vessel A and vessel B make the quota leasing agreement (x_1, x_2), changing vessel A's quota portfolio from $(Q_1^{\mathrm{A}}, Q_2^{\mathrm{A}})$ to $(Q_1^{\mathrm{A}} + x_1, Q_2^{\mathrm{A}} + x_2)$. Vessel A then knows

- Its optimal catch composition parameter c_{A}°,
- The corresponding optimal catch, $((t_1^{\mathrm{A}})^{\circ}, (t_2^{\mathrm{A}})^{\circ})$, and
- The associated maximal income I_{A}°.

In fact, depending on the three cases,

$$\text{Case I:} \qquad \frac{Q_2^{\mathrm{A}} + x_2}{Q_1^{\mathrm{A}} + x_1} < \frac{l_{\mathrm{A}}}{1 - l_{\mathrm{A}}}$$

$$\text{Case II:} \qquad \frac{l_{\mathrm{A}}}{1 - l_{\mathrm{A}}} \leqslant \frac{Q_2^{\mathrm{A}} + x_2}{Q_1^{\mathrm{A}} + x_1} \leqslant \frac{u_{\mathrm{A}}}{1 - u_{\mathrm{A}}}$$

$$\text{Case III:} \qquad \frac{u_{\mathrm{A}}}{1 - u_{\mathrm{A}}} < \frac{Q_2^{\mathrm{A}} + x_2}{Q_1^{\mathrm{A}} + x_1}$$

one has the following formulae:

$$c_{\mathrm{A}}^{\circ} = \begin{cases} l_{\mathrm{A}} & \text{(case I)} \\ \dfrac{Q_2^{\mathrm{A}} + x_2}{Q_1^{\mathrm{A}} + x_1 + Q_2^{\mathrm{A}} + x_2} & \text{(case II)} \\ u_{\mathrm{A}} & \text{(case III)} \end{cases}$$

$$((t_1^{\mathrm{A}})^{\circ}, (t_2^{\mathrm{A}})^{\circ}) = \begin{cases} \left(\dfrac{1 - l_{\mathrm{A}}}{l_{\mathrm{A}}}(Q_2^{\mathrm{A}} + x_2), Q_2^{\mathrm{A}} + x_2\right) & \text{(case I)} \\ (Q_1^{\mathrm{A}} + x_1, Q_2^{\mathrm{A}} + x_2) & \text{(case II)} \\ \left(Q_1^{\mathrm{A}} + x_1, \dfrac{u_{\mathrm{A}}}{1 - u_{\mathrm{A}}}(Q_1^{\mathrm{A}} + x_1)\right) & \text{(case III)} \end{cases}$$

$$I_{\mathrm{A}}^{\circ} = \begin{cases} \left(p_1 \dfrac{1 - l_{\mathrm{A}}}{l_{\mathrm{A}}} + p_2\right)(Q_2^{\mathrm{A}} + x_2) - q_1 x_1 - q_2 x_2 & \text{(case I)} \\ p_1 (Q_1^{\mathrm{A}} + x_1) + p_2 (Q_2^{\mathrm{A}} + x_2) - q_1 x_1 - q_2 x_2 & \text{(case II)} \\ \left(p_1 + p_2 \dfrac{u_{\mathrm{A}}}{1 - u_{\mathrm{A}}}\right)(Q_1^{\mathrm{A}} + x_1) - q_1 x_1 - q_2 x_2 & \text{(case III)} \end{cases}$$

Note that c_{A}°, $(t_1^{\mathrm{A}})^{\circ}$, $(t_2^{\mathrm{A}})^{\circ}$, and I_{A}° are functions of the variables (x_1, x_2).

Hence, for any choice of quota leasing agreement (x_1, x_2), vessel A's new quota portfolio $(Q_1^{\mathrm{A}} + x_1, Q_2^{\mathrm{A}} + x_2)$ results in the maximal income $I_{\mathrm{A}}^{\circ}(x_1, x_2)$—provided, of course, that it fishes optimally, i.e. according to the optimal catch composition parameter $c_{\mathrm{A}}^{\circ}(x_1, x_2)$. Thus, seen from the viewpoint of vessel A, the optimal quota leasing agreement (x_1, x_2) is the one that maximizes the function $I_{\mathrm{A}}^{\circ}(x_1, x_2)$.

We shall return to the following example throughout the rest of this paper.

3.4 Example. The market for cods and haddocks are assumed as follows.

Market Data	
Cod market price	$p_1 = 20$
Haddock market price	$p_2 = 14$
Cod quota leasing price	$q_1 = 10$
Haddock quota leasing price	$q_2 = 3$

The Scottish fleet's total cod and haddock quotas are 11000 and 21000 tonnes, respectively. Suppose that half of the fleet (vessel A) can vary its cod/haddock catch combination in the range from 50%/50% to 40%/60%, and that the other half (vessel B) in the range from 80%/20% to 20%/80%. Thus one has:

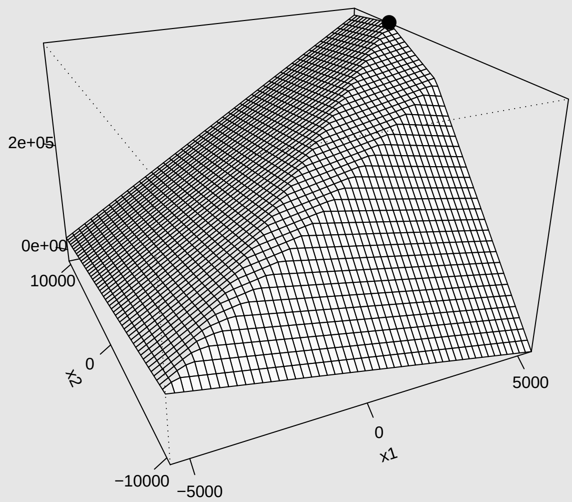
Vessel A Data	
Cod quota	$Q_1^{\mathrm{A}} = 5500$
Haddock quota	$Q_2^{\mathrm{A}} = 10500$
Lower catch composition bound	$l_{\mathrm{A}} = 0.5$
Upper catch composition bound	$u_{\mathrm{A}} = 0.6$

Vessel B Data	
Cod quota	$Q_1^{\mathrm{B}} = 5500$
Haddock quota	$Q_2^{\mathrm{B}} = 10500$
Lower catch composition bound	$l_{\mathrm{B}} = 0.2$
Upper catch composition bound	$u_{\mathrm{B}} = 0.8$

If the vessels choose not to lease quotas from each other—corresponding to the quota leasing agreement $(x_1, x_2) = (0, 0)$—then we may consider them as isolated. In this case, our solution to the single vessel problem 2.9 shows that their maximal incomes are:

$$I_{\mathrm{A}}^{\circ} = 225500 \qquad \text{and} \qquad I_{\mathrm{B}}^{\circ} = 257000$$

As seen above, the given constants $Q_1^{\mathrm{A}}, Q_2^{\mathrm{A}}, Q_1^{\mathrm{B}}, Q_2^{\mathrm{B}}, l_{\mathrm{A}}, u_{\mathrm{A}}$ completely describe vessel A's maximal income $I_{\mathrm{A}}^{\circ}(x_1, x_2)$ as a function of the quota leasing agreement (x_1, x_2) made by vessels A and B. Its graph is depicted below.



Graph of income function $I_{\mathrm{A}}^{\circ}(x_1, x_2)$ for vessel A

It follows that the optimal quota leasing agreement *for vessel A* is $(x_1^{\circ}, x_2^{\circ}) = (5500, 6000)$ (meaning that vessel A should try to lease 5500 tonnes of cod quota and 6000 tonnes of haddock quota from vessel B) resulting in the new quota portfolio $(Q_1^{\mathrm{A}} + x_1, Q_2^{\mathrm{A}} + x_2) = (11000, 16500)$ and a larger maximal income:

$$I_{\mathrm{A}}^{\circ}(5500, 6000) = 378000.$$

Example (continued). A couple of observations are in order here:

- If vessel A were given the quota portfolio $(11000, 16500)$ to begin with, its maximal income would have been 451000, cf. Example 2.10. In the present situation, vessel A must pay $5500 \cdot 10 + 6000 \cdot 3 = 73000$ to obtain the quota portfolio, thus reducing its maximal income to $451000 - 73000 = 378000$.
- To obtain the larger (than 225500) maximal income of 378000, it is required that vessel A fishes optimally, that is, according to the optimal catch composition parameter $c_{\mathrm{A}}^{\circ}(5500, 6000) = 0.6$, corresponding to a cod/haddock catch composition of 40%/60%. In this case, vessel A's fishery stops after having caught

$$(t_1^{\mathrm{A}})^{\circ}(5500, 6000) = 11000 \text{ tonnes of cod, and}$$
$$(t_2^{\mathrm{A}})^{\circ}(5500, 6000) = 16500 \text{ tonnes of haddock,}$$

  that is, both quotas are reached simultaneously.

Of course, the quota leasing agreement $(x_1^{\circ}, x_2^{\circ}) = (5500, 6000)$, which is optimal for vessel A, can never become a reality as it reduces vessel B's cod quota to zero and thus stops vessel B's fishery.

The example above illustrates that we must consider the situation from the viewpoint of vessel A and vessel B simultaneously. Analogously to how we found the functions c_{A}°, $(t_1^{\mathrm{A}})^{\circ}$, $(t_2^{\mathrm{A}})^{\circ}$, and I_{A}° for vessel A, we can find the corresponding functions for vessel B. This is done below.

3.5 Quota Leasing Agreements from the Viewpoint of Vessel B. Suppose that vessel A and vessel B make the quota leasing agreement (x_1, x_2). Depending on the following three cases,

$$\text{Case I:} \qquad \frac{Q_2^{\mathrm{B}} - x_2}{Q_1^{\mathrm{B}} - x_1} < \frac{l_{\mathrm{B}}}{1 - l_{\mathrm{B}}}$$

$$\text{Case II:} \qquad \frac{l_{\mathrm{B}}}{1 - l_{\mathrm{B}}} \leqslant \frac{Q_2^{\mathrm{B}} - x_2}{Q_1^{\mathrm{B}} - x_1} \leqslant \frac{u_{\mathrm{B}}}{1 - u_{\mathrm{B}}}$$

$$\text{Case III:} \qquad \frac{u_{\mathrm{B}}}{1 - u_{\mathrm{B}}} < \frac{Q_2^{\mathrm{B}} - x_2}{Q_1^{\mathrm{B}} - x_1}$$

vessel B's optimal catch composition parameter, its optimal catch, and the associated maximal income are given by the following formulae.

$$c_{\mathrm{B}}^{\circ} = \begin{cases} l_{\mathrm{B}} & \text{(case I)} \\ \dfrac{Q_2^{\mathrm{B}} - x_2}{Q_1^{\mathrm{B}} - x_1 + Q_2^{\mathrm{B}} - x_2} & \text{(case II)} \\ u_{\mathrm{B}} & \text{(case III)} \end{cases}$$

$$((t_1^{\mathrm{B}})^{\circ}, (t_2^{\mathrm{B}})^{\circ}) = \begin{cases} \left(\dfrac{1 - l_{\mathrm{B}}}{l_{\mathrm{B}}}(Q_2^{\mathrm{B}} - x_2), Q_2^{\mathrm{B}} - x_2\right) & \text{(case I)} \\ (Q_1^{\mathrm{B}} - x_1, Q_2^{\mathrm{B}} - x_2) & \text{(case II)} \\ \left(Q_1^{\mathrm{B}} - x_1, \dfrac{u_{\mathrm{B}}}{1 - u_{\mathrm{B}}}(Q_1^{\mathrm{B}} - x_1)\right) & \text{(case III)} \end{cases}$$

$$I_{\mathrm{B}}^{\circ} = \begin{cases} \left(p_1 \dfrac{1 - l_{\mathrm{B}}}{l_{\mathrm{B}}} + p_2\right)(Q_2^{\mathrm{B}} - x_2) + q_1 x_1 + q_2 x_2 & \text{(case I)} \\ p_1(Q_1^{\mathrm{B}} - x_1) + p_2(Q_2^{\mathrm{B}} - x_2) + q_1 x_1 + q_2 x_2 & \text{(case II)} \\ \left(p_1 + p_2 \dfrac{u_{\mathrm{B}}}{1 - u_{\mathrm{B}}}\right)(Q_1^{\mathrm{B}} - x_1) + q_1 x_1 + q_2 x_2 & \text{(case III)} \end{cases}$$

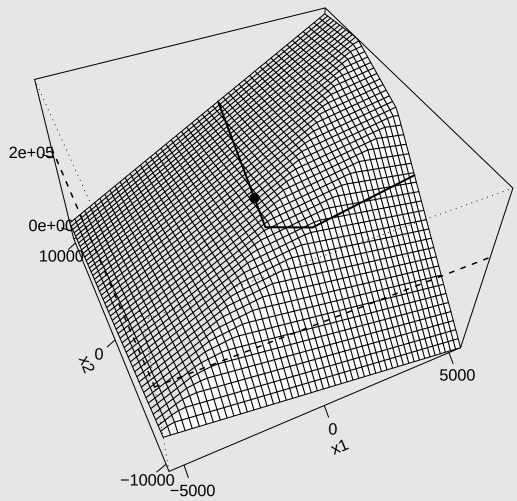
If vessels A and B decide not to exchange any quotas—corresponding to the quota leasing agreement $(x_1, x_2) = (0, 0)$—then their maximal incomes are $I_{\mathrm{A}}^{\circ}(0, 0)$ and $I_{\mathrm{B}}^{\circ}(0, 0)$, respectively. Obviously, vessel A is only interested in making a quota leasing agreement (x_1, x_2) if it is profitable, that is, if it results in a larger maximal income than $I_{\mathrm{A}}^{\circ}(0, 0)$. Similarly for vessel B. Thus, the *profitable* quota leasing agreements for vessels A and B are the subsets of the square $[-Q_1^{\mathrm{A}}, Q_1^{\mathrm{B}}] \times [-Q_2^{\mathrm{A}}, Q_2^{\mathrm{B}}]$ given by

$$\mathcal{P}_{\mathrm{A}} = \big\{ (x_1,x_2) \;\big|\; I_{\mathrm{A}}^{\circ}(x_1,x_2) \geqslant I_{\mathrm{A}}^{\circ}(0,0) \big\}, \text{ and}$$
$$\mathcal{P}_{\mathrm{B}} = \big\{ (x_1,x_2) \;\big|\; I_{\mathrm{B}}^{\circ}(x_1,x_2) \geqslant I_{\mathrm{B}}^{\circ}(0,0) \big\}.$$

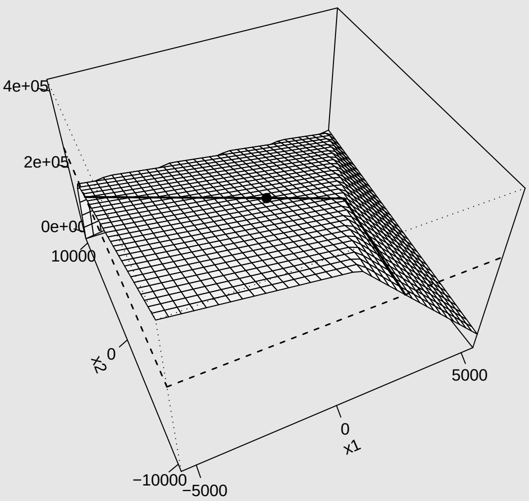
A necessary condition for vessels A and B to make a quota leasing agreement (x_1,x_2) is that this agreement is profitable for both parties, that is, (x_1,x_2) must belong to the intersection $\mathcal{P} = \mathcal{P}_{\mathrm{A}} \cap \mathcal{P}_{\mathrm{B}}$.

It is of interest for both vessels to find the set $\mathcal{P}$ of quota leasing agreements which are profitable for both of them. As illustrated by Example 3.8, there might not be any(!), however, usually there are many. In Example 3.4, vessel B is superior to vessel A in the sense that both vessels have the same number of quotas, but vessel B is more flexible in its catch composition (which ranges from 80%/20% to 20%/80%) than vessel A (which only ranges from 50%/50% to 40%/60%). However, even in this case there are many quota leasing agreements that are profitable for both vessels. This is explored in Examples 3.6 and 3.7 below.

3.6 Example. Consider the setup in Example 3.4. The graphs of vessel A's and vessel B's income functions $I_{\mathrm{A}}^{\circ}(x_1,x_2)$ and $I_{\mathrm{B}}^{\circ}(x_1,x_2)$ are illustrated below.

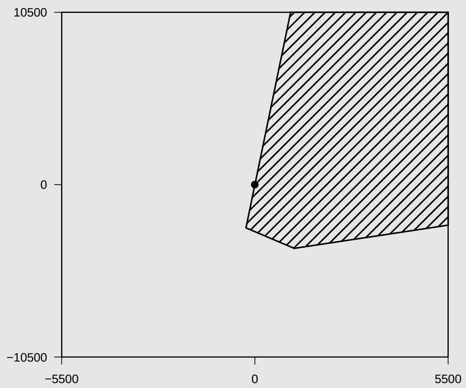


Graph of income function $I_{\mathrm{A}}^{\circ}(x_1,x_2)$ for vessel A

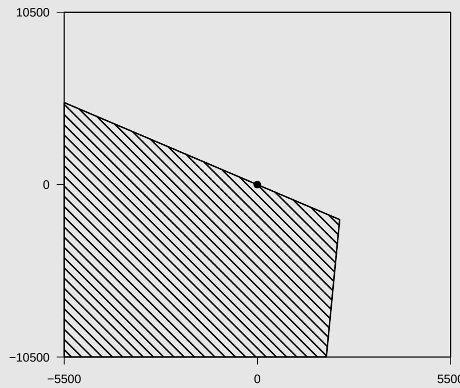


Graph of income function $I_{\mathrm{B}}^{\circ}(x_1,x_2)$ for vessel B

From these graphs one finds the sets of profitable quota leasing agreements for vessels A and B:

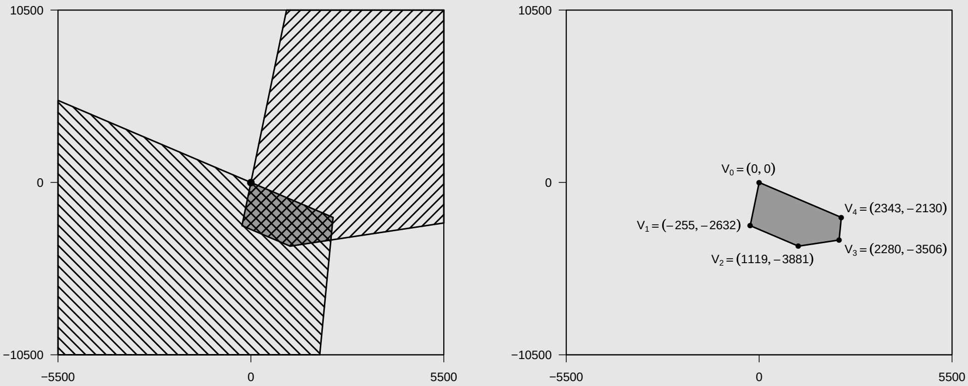


The set $\mathcal{P}_{\mathrm{A}}$ of profitable quota leasing agreements for vessel A



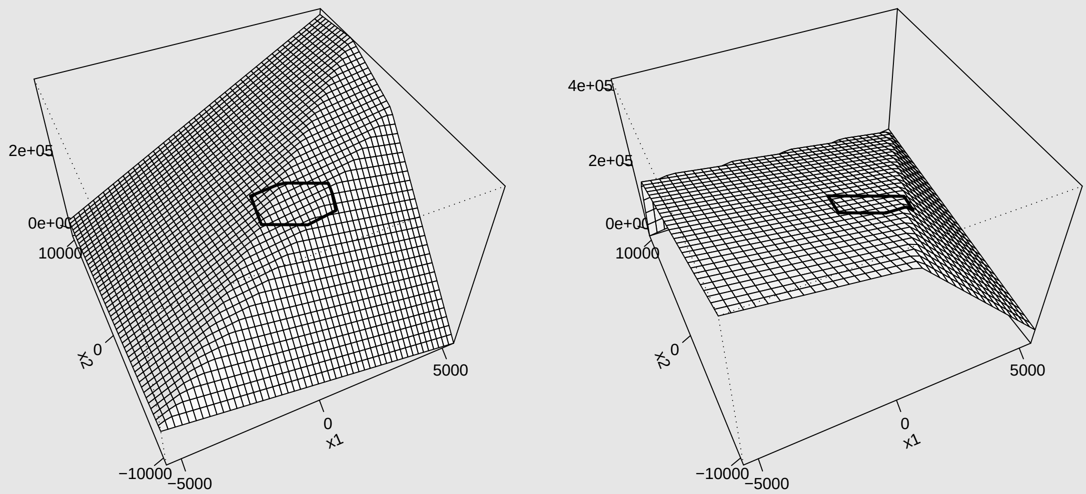
The set $\mathcal{P}_{\mathrm{B}}$ of profitable quota leasing agreements for vessel B

Example (continued). The set $\mathcal{P}$ of quota leasing agreements that are profitable for both vessels is found by intersecting the two sets $\mathcal{P}_{\mathrm{A}}$ and $\mathcal{P}_{\mathrm{B}}$, as illustrated below.



The set $\mathcal{P}$ of quota leasing agreements that are profitable for both vessels

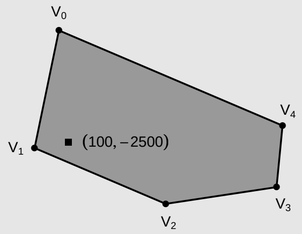
We see that $\mathcal{P}$ is a polygon with vertices $V_0,\dots,V_4$ whose coordinates are shown in the figure above. The corresponding areas on vessel A's and B' income functions are:



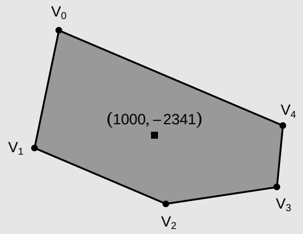
In general, the set $\mathcal{P}$ of quota leasing agreements that are profitable for both vessels is a polygon. Of course, it is possible to write down explicit formulae for the vertices $V_0,V_1,\dots$ of this polygon in terms of the given market data p_1,p_2,q_1,q_2, and vessel data $Q_1^{\mathrm{A}},Q_2^{\mathrm{A}},l_{\mathrm{A}},u_{\mathrm{A}}$ and $Q_1^{\mathrm{B}},Q_2^{\mathrm{B}},l_{\mathrm{B}},u_{\mathrm{B}}$, however, since these formulae are rarther complicated, we shall not attempt to present them here. Instead, we note that in all specific examples, one can easily find the set $\mathcal{P}$, as it is done in Example 3.6.

Note that the quota leasing agreements (x_1,x_2) corresponding to boundary points, i.e. the edges, of the polygon $\mathcal{P}$ are not interesting, since either vessel A or vessel B will not make (or lose) any mony from such an agreement. Hence, the interesting quota leasing agreements correspond to interior points of the set $\mathcal{P}$.

3.7 Example. Consider the set $\mathcal{P}$ of quota leasing agreements that are profitable for both vessels, found in Example 3.6 above. The vertex $V_0 = (0,0)$ corresponds to the situation where the vessels lease no quotas from or to each other. We consider as well two other possible quota leasing agreements:



Quota leasing agreement $(x_1,x_2) = (100,-2500)$



Quota leasing agreement $(x_1,x_2) = (1000,-2341)$

Since $(x_1,x_2) = (100,-2500)$ (corresponding to the situation where vessel A leases 100 tonnes of cod quota *from*, and 2500 tonnes of haddock quota *to* vessel B) and $(x_1,x_2) = (1000,-2341)$ are points in $\mathcal{P}$, these quota leasing agreements are profitable for both vessels. However, as we shall see below, one vessel might benefit more than the other.

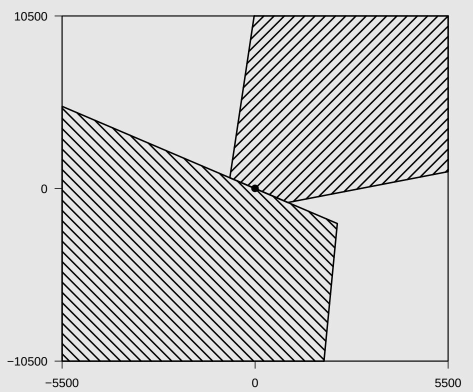
Computing the functions I_A°, I_B° and c_A°, c_B° at the points $(0,0)$, $(100,-2500)$, and $(1000,-2341)$ gives us the maximal income, and the catch composition required to obtain this, for each of the three situations.

(x_1,x_2)	**Vessel A**	**Vessel B**
$(0,0)$	Income $I_A^\circ = 225500$ Catch $c_A^\circ = 0.60$ (40%/60%)	Income $I_B^\circ = 257000$ Catch $c_B^\circ = 0.66$ (34%/66%)
$(100,-2500)$	Income $I_A^\circ = 230500$ Catch $c_A^\circ = 0.59$ (41%/59%)	Income $I_B^\circ = 283500$ Catch $c_B^\circ = 0.71$ (29%/71%)
$(1000,-2341)$	Income $I_A^\circ = 241250$ Catch $c_A^\circ = 0.56$ (44%/56%)	Income $I_B^\circ = 272750$ Catch $c_B^\circ = 0.74$ (26%/74%)

Compared to the situation $(0,0)$, where no quotas are leased, we see that if the quota leasing agreement $(100,-2500)$ is made then vessel A makes an additional 5000, whereas vessel B makes an additional 26500. If the quota leasing agreement $(1000,-2341)$ is made then both vessels make an additional 15750.

By the end of the day, whatever quota leasing agreement is made between the two vessels depends on which has the best negotiation skills.

3.8 Example. Consider the same setup as in Example 3.4, however, assume this time that vessel A's possible cod/haddock catch composition ranges from 40%/60% to 30%/70% (that is, $l_A = 0.6$ and $u_A = 0.7$). In this situation, the sets $\mathcal{P}_A$ and $\mathcal{P}_B$ have only an edge in commen, and hence there are *no* quota leasing agreements which are profitable for both vessels.



4. Conclusion

In a mixed fishery managed by catch quotas (as opposed to landing quotas), a vessel must stop fishing as soon as one of the quotas in its quota portfolio has been reached. A vessel may use its equipment to influence its catch composition within a certain range, and thereby try to optimize its catch in order to maximize the income (market prices for the types of fish involved in the mixed fishery are given). A priori, a vessel's quota portfolio is given however, it is possible to lease quotas from other vessels in order to obtain a more desirable portfolio. Therefore, a vessel fishing under catch quota management has several ways to optimize its catch and income. In this paper, we have given precise mathematical formulations of the relevant optimization problems, and solved them.

References

[1] Proposal for a Regulation of the European parliament and of the Council on the Common Fisheries Policy. Brussels 13th July 2011 COM(2011) 425 Final.